ON A PERTURBATION METHOD IN THE SHORT-BLOW PROBLEM

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A. I. DERZHAVINA and O. S. RYZHOV
(Moscow)
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The problem is considered of the motion of an initially cold gas that is close to the motion resulting from the action of a short-time blow. The basic self-similar solution found by the authors of [1], [2] and [3] is subjected to small superimposed perturbations, after which linearization is carried out on the Euler equations and the Hugoniot conditions at the shock-wave front. The system of equations for the variation of the parameters of the gas is reduced to one second-order differential equation, which in the case of a diatomic gas with ratio of specific heats $\varkappa = {}^7/_5$ is transformed into the hypergeometric equation of Gauss. The solution of the problem for the variations permits a proper analytic continuation of the unknown functions into the region bounded by a vacuum, and finding the location of the boundary itself. In this way is eliminated the well-known difficulty connected with the divergence of the energy integral in the basic self-similar solution.

1. The problem of the motion of gas under the action of a short-time blow was set by Zel dovich [1] in the following form. Let a half-space bounded by a vacuum be filled with perfect gas having a ratio of specific heats $\varkappa$ At the initial instant the gas is quiescent, its density is constant, and the temperature and pressure are equal to zero. A strong short-time impulse of pressure is applied at the boundary of the gas. It is required to determine the resulting motion for sufficiently large time after the moment when the impulse ceased to act.

It is evident that a shock wave will propagate through the cold gas. Since the other boundary of the flow is a vacuum, the intensity of the shock wave falls with time in the maximum possible way. The solution of the formulated problem is self-similar, and for $\varkappa = {}^7/_5$ its exact form is established in the works [4 - 6].

Let us now suppose that small perturbations are superimposed on the basic motion and in order to simplify the mathematical investigation, let us assume at the beginning that $\varkappa = {}^7/_5$.

If t denotes the time, x the coordinate, v the speed, ρ the density and p the pressure, then under the conditions described above the Euler equations are written in the form [7]

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{1.1}$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \frac{7}{5}p\frac{\partial v}{\partial x} = 0$$

The density ρ_1 of the quiescent gas is assumed constant. As for the intensity of the wave generated by the action of the short-time impulse, we assume it sufficiently strong that, unless the contrary is specifically stated, we neglect the pressure p_1 ahead of the surface of discontinuity. Then for the transition across the shock front, moving with speed $c(t)$, the following relations should be satisfied [7]

$$v_2 = {}^5/_6 c, \qquad \rho_2 = 6\rho_1, \qquad p_2 = {}^5/_6 \rho_1 c^2 \tag{1.2}$$

Subscript 2 indicates the gas in the shocked state. Besides the Hugoniot conditions (1.2), in order to construct a solution of the system of differential equations (1.1) it is still necessary to satisfy one more boundary condition, which states that the pressure and density of the gas vanish at the boundary with the vacuum.

It is convenient to formulate the problem for the variations as an inverse one, writing for the coordinate of the shock-wave front

$$x_s = (At)^{3/5}(1 - \varepsilon t^{-m})$$

Here ε is a small parameter, the constant A is related to the characteristic pressure impulse, but remains undetermined, and the power exponent $m > 0$ under the conditions that the counter-pressure is considered negligibly small. Transforming to new independent variables t and $\lambda = x(At)^{-3/5}$, we represent the unknown functions as

$$v = v_{20}[f(\lambda) + \varepsilon t^{-m} f_m(\lambda)], \qquad \rho = \rho_{20}[g(\lambda) + \varepsilon t^{-m} g_m(\lambda)]$$
$$p = p_{20}[h(\lambda) + \varepsilon t^{-m} h_m(\lambda)] \tag{1.3}$$

In Eqs. (1.3) the quantities $v_{20}(t)$ and $p_{20}(t)$ are determined by the conditions (1.2) if we assume that

$$c = {}^3/_5 A^{3/5} t^{-2/5}$$

and the density ρ_{20} is constant and equal to $6\rho_1$. As for the functions $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ they are evidently solutions of the short-blow problem [1, 2]. Corresponding to the expansions (1.3) we write the equation of the shock-wave front in the form

$$\lambda_s = 1 - \varepsilon t^{-m}$$

In accord with the standard procedure of perturbation methods, the initial conditions for the unknown functions f_m, g_m and h_m are to be imposed at the point $\lambda = 1$. Retaining in all relations only terms of first order in ε and neglecting terms having a higher order of smallness, we have at $\lambda = 1$

$$f_m = \frac{5}{3}m - 1 + \frac{df}{d\lambda}, \qquad g_m = \frac{dg}{d\lambda}, \qquad h_m = 2\left(\frac{5}{3}m - 1\right) + \frac{dh}{d\lambda} \tag{1.4}$$

Linearization of the equations of motion (1.1) is carried out completely analogously, as a result of which we deduce the homogeneous system

$$g\frac{df_m}{d\lambda} + \left(f - \frac{6}{5}\lambda\right)\frac{dg_m}{d\lambda} + \frac{dg}{d\lambda}f_m + \left(\frac{df}{d\lambda} - 2m\right)g_m = 0 \tag{1.5}$$

$$\left(f - \frac{6}{5}\lambda\right)g\frac{df_m}{d\lambda} + \frac{1}{5}\frac{dh_m}{d\lambda} + \left[\frac{df}{d\lambda} - 2\left(\frac{2}{5} + m\right)\right]gf_m + \left[\frac{df}{d\lambda}\left(f - \frac{6}{5}\lambda\right) - \frac{4}{5}f\right]g_m = 0$$

$$h\frac{df_m}{d\lambda} + \frac{5}{7}\left(f - \frac{6}{5}\lambda\right)\frac{dh_m}{d\lambda} + \frac{5}{7}\frac{dh}{d\lambda}f_m + \left[\frac{df}{d\lambda} - \frac{10}{7}\left(\frac{4}{5} + m\right)\right]h_m = 0$$

This system determines the functions f_m, g_m, h_m in the semi-infinite interval $-\infty < \lambda < 1$.

It is immediately possible to give some exact solutions of the Cauchy problem (1.4) for Eqs. (1.5), based on the group properties of the short-blow problem. As is known, its self-similarity is connected with the existence of a certain group of similarity transformations. A shift in the constant A leads once again to a solution of the short-blow problem with a certain change in the value of this parameter. In addition, the original Euler equations and Hugoniot conditions are invariant with respect to a shift in the time t and the coordinate x. Taking these remarks into consideration, we have

$$f_m = -f + \lambda \frac{df}{d\lambda}, \quad g_m = \lambda \frac{dg}{d\lambda}, \quad h_m = -2h + \lambda \frac{dh}{d\lambda} \quad \text{for } m = 0$$
$$f_m = \frac{2}{3} f + \lambda \frac{df}{d\lambda}, \quad g_m = \lambda \frac{dg}{d\lambda}, \quad h_m = \frac{4}{3} h + \lambda \frac{dh}{d\lambda} \quad \text{for } m = 1$$
$$f_m = \frac{df}{d\lambda}, \quad g_m = \frac{dg}{d\lambda}, \quad h_m = \frac{dh}{d\lambda} \quad \text{for } m = \frac{3}{5} \tag{1.6}$$

2. In [8] a closed-form integral was given for the equations of one-dimensional self-similar gas motion, being a consequence of the laws of conservation of entropy and particles. Later the author of [9] established an integral of adiabaticity for the equations of variations that are taken relative to a self-similar solution. Taking advantage of the developments in [8, 9] we find that it is possible to deduce a first integral of the system of equations (1.5), which has the form

$$\frac{5f_m}{5f - 6\lambda} + \frac{35m - 1}{20} \frac{g_m}{g} + \frac{3 - 5m}{4} \frac{h_m}{h} = Cg^{-7m/4} h^{5m/4} \tag{2.1}$$

The constant C appearing here is determined from the Cauchy data (1.4), and as a result

$$C = {}^5/_6 (m - 1)(3 - 5m)$$

For the subsequent analysis it is convenient to transform the system of three equations (1.5) into one equation for the function f_m. The result is of second and not third order if equation (2.1) is considered in the process of transformation. To simplify the calculations we use at once the relations

$$f = 2\lambda - 1, \qquad g = (5-4\lambda)^{-5/2}, \qquad h = (5 - 4\lambda)^{-3/2}$$

which give the exact solution of the sharp-blow problem under the assumption that the gas is diatomic [4 - 6]. After changing to the new independent variable $\xi = (5-4\lambda)/7$ we obtain for the function f_m the hypergeometric equation of Gauss

$$\xi(1 - \xi) \frac{d^2 f_m}{d\xi^2} + [\gamma - (1 + \alpha + \beta)\xi] \frac{df_m}{d\xi} - \alpha\beta f_m = 0 \tag{2.2}$$
$$\alpha = -{}^5/_2 m, \qquad \beta = {}^3/_2 - {}^5/_2 m, \qquad \gamma = \alpha - \beta = -{}^3/_2 \tag{2.3}$$

The right-hand side of Eq. (2.2) is equal to zero, so that the value of function f_m does not depend upon the constant C. The form of the function g_m is determined by the formula

$$g_m = \frac{10}{5m + 2} (7\xi)^{-7/2} \left\{ \frac{2}{7} C (7\xi)^{(5m+2)/2} + \frac{1}{2} [5 + (3 - 5m)\xi] f_m + \xi(\xi - 1) \frac{df_m}{d\xi} \right\}$$

As for the function h_m, it is found from the integral of adiabaticity (2.1). Both the latter quantities g_m and h_m change with changes in the constant in its right-hand side.

For $\lambda = 1$ the variable $\xi = 1/7$, that is the self-similar coordinate of the shock front in the short-blow problem corresponds to a regular point of Eq. (2.2). The conditions (1.4) arising from the laws of conservation of the flux of mass, momentum, and energy of matter in the transition across the surface of strong discontinuity permit a Cauchy problem to be formulated for the equation that has been found, and by the same token a solution to be determined uniquely in the range $1/7 \leqslant \xi < 1$. Namely, for $\xi = 1/7$ we have

$$f_m = 1/3\,(5m+3), \qquad df_m/d\xi = {}^{35}/_{12}m\,(3-5m) \tag{2.4}$$

The point $\xi = 1$ is singular for the hypergeometric equation; in the case under consideration it gives the location of the limiting characteristic in the original self-similar solution. In fact, the speed of sound is

$$a = \sqrt{\varkappa p/\rho} = {}^1/_{10}\,\sqrt{7}\,A^{1/5}t^{-3/5}(5-4\lambda)^{1/2}$$

Hence the equation of the characteristic $x = x_c$ in the solution of the short-blow problem is

$$dx_c/dt = {}^1/_2 A^{1/5}t^{-3/5}\,[2\lambda - 1 + {}^1/_5\sqrt{7}\,(5-4\lambda)^{1/2}]$$

If we seek a particular integral of this equation in the form

$$x_c = \lambda_c A^{1/5}t^{2/5}$$

then the values $5/4$ and $-1/2$ are obtained for the constant λ_c. The first of these is to be discarded, since it leaves the region of determination of the self-similar solution, and the second value gives just $\xi = 1$. The continuation of the solution of the problem of variations through this point should remain regular, since the propagation of any sort of disturbance does not take place along the limiting characteristic. Thus the basic question consists in elucidating the behavior of the functions f_m, g_m and h_m in the vicinity of the point $\xi = 1$. The answer depends essentially on what are the values of the parameters α and β in Eqs. (2.3).

We assume at first that the difference $\gamma - (\alpha + \beta) = 5m - 3$ is not a whole number; then the solution of Eq. (2.2) can be given in the form [10]

$$f_m = c_1 F\,(\alpha, \beta; \alpha+\beta-\gamma+1; 1-\xi) + c_2\,(1-\xi)^{5m-3}\,F\,(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-\xi) \tag{2.5}$$

Here, as usual, the letter F denotes the hypergeometric function. Recalling Eqs. (2.3) and setting $2 - 5m/2 = b$, we have

$$\alpha = b-2, \quad b = \beta - {}^1/_2, \quad \alpha+\beta-\gamma+1 = 2b$$

We now use the known relations [10]

$$F\,(b, b-{}^1/_2; 2b; z) = ({}^1/_2 + {}^1/_2\sqrt{1-z})^{1-2b}$$

$$F\,(\alpha-n, \beta; \gamma; z) = \frac{1}{(\gamma-\alpha)_n}\, z^{1+\alpha-\gamma}\,(1-z)^{\gamma+n-\alpha-\beta} \times \frac{d^n}{dz^n}\,[z^{\gamma+n-\alpha-1}(1-z)^{\alpha+\beta-\gamma}F\,(\alpha, \beta; \gamma; z)]$$

$$(\gamma-\alpha)_n = (\gamma-\alpha)(\gamma-\alpha+1)(\gamma-\alpha+2)\ldots(\gamma-\alpha+n-1)$$

to transform the first of the hypergeometric functions appearing in the right-hand side of Eq. (2.5). As a result we find

$$F\,(\alpha, \beta; \alpha+\beta-\gamma+1; 1-\xi) = \frac{2^{5(1-m)}\xi^{3/2}(1-\xi)^{(5m-2)/2}}{(4-5m)(6-5m)}\,\frac{d^2}{d\xi^2}\,\frac{(1-\xi)^{(6-5m)/2}(1+\sqrt{\xi})^{5m-3}}{\sqrt{\xi}} \tag{2.6}$$

Now let $b = 5m/2 - 1$. Then

$$\gamma-\alpha = b-2, \quad \gamma-\beta = b-{}^1/_2, \quad \gamma-\alpha-\beta+1 = 2b$$

and the second hypergeometric function is transformed completely analogously:

$$F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-\xi)=$$
$$=\frac{2^{5m-1}\xi^{3/2}(1-\xi)^{-5m/2}}{5m(5m-2)}\frac{d^2}{d\xi^2}\frac{(1-\xi)^{5m/2}(1+\sqrt{\xi})^{3-5m}}{\sqrt{\xi}} \tag{2.7}$$

Equations (2.6) and (2.7) show that in the case under consideration the solution of the problem of variations is expressed in terms of elementary functions. This property significantly simplifies the subsequent analysis. We substitute the representation (2.5) into the boundary conditions (2.4) and determine the values of the constants c_1 and c_2; both of them evidently depend upon the power exponent m. In order that the parameters of the gas not have singularities on the limiting characteristic $\xi = 1$ it is necessary to set $c_2 = 0$ This condition gives

$$\left[\frac{35m(3-5m)}{4}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-\xi)-\right.$$
$$\left.-(3+5m)\times\frac{d}{d\xi}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-\xi)\right]_{\xi=1/7}=0$$

As was just now established, the hypergeometric functions appearing here are expressed in elementary terms by Eq. (2.6). As a result, the quantity m must satisfy a fourth-order algebraic equation

$$a_4m_*^4+a_3m_*^3+a_2m_*^2+a_1m_*=0, \qquad m_*=5m \tag{2.8}$$

with coefficients

$$a_1=4(1062\sqrt{7}+2709), \qquad a_2=2(1503\sqrt{7}+3906)$$
$$a_3=-(677\sqrt{7}+1799), \qquad a_4=7(7\sqrt{7}+19)$$

The first root of this equation is obvious: $m_1 = 0$. It corresponds to the invariance of the original self-similar solution with respect to a displacement of the constant A. With the use of Eqs. (1.6) we determine the unknown functions

$$f_0=1, \qquad g_0=10\lambda(5-4\lambda)^{-7/2}, \qquad h_0=(14\lambda-10)(5-4\lambda)^{-5/2}$$

The invariance of the self-similar solution with respect to a displacement of the coordinate x gives at once the possibility of finding the second root of Eq. (2.8). We have $m_2 = 3/5$, and

$$f_{3/5}=2, \qquad g_{3/5}=10(5-4\lambda)^{-7/2}, \qquad h_{3/5}=6(5-4\lambda)^{-5/2}$$

The remaining two roots are easily determined as the solutions of a quadratic equation, with

$$m_3=\frac{6}{5}, \qquad m_4=\frac{1}{5}\,\frac{236\sqrt{7}+602}{49\sqrt{7}+133}\approx 0.933895 \tag{2.9}$$

It is essential that the general approach considered, consisting in setting the constant c_2 equal to zero, gave only a finite set of values of m. For the first three of them, the difference $\gamma-(\alpha+\beta)$ is equal to -3, 0 and $+3$, respectively; that is, in spite of the assumption made above, they are whole numbers. Only for the fourth root m_4 is the difference $\gamma-(\alpha+\beta)$ in fact expressed as an infinite fraction.

As a rule, at the singular points of a hypergeometric equation one of its linearly independent solutions is regular, and the other has a singularity. However, the choice of the parameter m may be achieved such that both of them are represented by analytic functions in the vicinity of the point $\xi = 1$. In this case the perturbations also will not propagate along the limiting characteristic.

It is natural that this choice of m necessarily gives integral values of the difference $\gamma-(\alpha+\beta)$. We set at first $m = (2N+3)/5$ with $N = 1, 2, 3, \ldots$. Then for the linearly independent integrals f_m^1 and f_m^2 of Eq. (2.2) the following equations hold:

$$f_m{}^1 = F\left(-N_2 - N - \frac{3}{2}; -2N+1; 1-\xi\right) = \frac{2^N N!}{(k_1+1)_N (\zeta+1)^N} P_N^{(k_1,k_2)}(\zeta)$$

$$f_m{}^2 = \xi^{(2N+3)/2} F\left(-N+1, -N-\frac{3}{2}; -\frac{1}{2}; \frac{1}{\xi}\right) = \tag{2.10}$$

$$= \frac{(-1)^{N-1}(N-1)!\, 2^{(2N+3)/2}}{(k_2+1)_{N-1}(\zeta+1)^{(2N+3)/2}} P_{N-1}^{(k_1,k_2)}(\zeta)$$

$$(k_1 = -2N, \quad k_2 = {}^3/_2, \quad \zeta = (2-\xi)/\xi)$$

Here the symbol $P_N^{(k_1,k_2)}$ serves as the standard notation for the Jacobi polynomials. For $m = 2N/5$ with $N = 4, 5, 6..$ we have analogously

$$f_m{}^1 = F\left(-N_3 - N + \frac{3}{2}; -2N+4; 1-\xi\right) = \frac{2^N N!}{(k_3+1)_N (\zeta+1)^N} P_N^{(k_3,k_4)}(\zeta)$$

$$f_m{}^2 = \xi^{(2N-3)/2} F\left(-N+\frac{3}{2}, -N+4; \frac{5}{2}; \frac{1}{\xi}\right) = \tag{2.11}$$

$$= \frac{(-1)^N 2^{(2N-3)/2}(N-4)!}{(k_4+1)_{N-4}(\zeta+1)^{(2N-3)/2}} P_{N-4}^{(k_3,k_4)}(\zeta)$$

$$(k_3 = -2N+3, \quad k_4 = -{}^3/_2)$$

The relations (2.10) and (2.11) solve the given problem, since the gas flow constructed with their help is free of any singularity on the limiting characteristic. The combination $c_1 f_m{}^1 + c_2 f_m{}^2$ of the functions $f_m{}^1$ and $f_m{}^2$ can satisfy the initial data (2.4) if the constants c_1 and c_2 are properly determined. It is clear that for $N = 1$ the constant $c_2 = 0$. In fact, in this case $m = 1$, so that the quantity $f_m{}^1$ from (2.10) can also be obtained from the condition of invariance of the self-similar solution with respect to a displacement in time. Turning to Eqs. (1.6), we determine the form of the unknown functions

$$f_1 = {}^2/_3(5\lambda - 1), \quad g_1 = 10\lambda(5-4\lambda)^{-7/2}, \quad h_1 = {}^2/_3(10+\lambda)(5-4\lambda)^{-5/2}$$

3. The solutions of the perturbation problem with $m = 0, {}^3/_5$ and 1 can at once be excluded from consideration. Indeed, changing in a proper way the constant A, the origin of time and the coordinates, it is easy in the linear approximation to make them expansions in powers of the basic self-similar solution. Henceforth we will assume that this precedure has been accomplished. The smallest values of m are given by Eqs. (2.9), where a direct test shows that the formulas

$$f = {}^1/_9(1+2\lambda)^3, \quad g = -{}^5/_{28}(5-4\lambda)^{1/2} + {}^5/_{252}(5-4\lambda)^{-7/2}(1+2\lambda)^3(25-6\lambda)$$

$$h = {}^5/_{28}(5-4\lambda)^{3/2} + {}^1/_{756}(1+2\lambda)^3(5-4\lambda)^{-5/2}(465-246\lambda)$$

are a representation of the unknown functions for $m = {}^6/_5$. They follow from Eq. (2.5) if in it we set $c_1 = 0$ and $c_2 = 3 \cdot 6^{-3} \cdot 7^3$.

The self-similar solution of the short-blow problem states that the gas instantly spreads through the whole initially empty half-space. In the vicinity of the infinitely remote point the behavior of its parameters expressed by that solution is incorrect; hence, as is known [1, 2], the energy integral is found to be divergent.

Overcoming this difficulty is associated with the proper analytic continuation of the solution of the Euler equations in the region of large negative values of the coordinate x. As $x \to -\infty$ also the similarity variable $\lambda \to -\infty$ if the time t is assumed fixed. Letting the absolute value of λ tend to infinity, we write asymptotic expansions for the functions f_m, g_m and h_m. For any value of the power-law exponent m we have

$$\begin{aligned} f_m &= A_1 (5 - 4\lambda)^{(5m-3)/2} + A_2 (5 - 4\lambda)^{5m/2} \\ g_m &= B_1 (5 - 4\lambda)^{5(m-2)/2} + B_2 (5 - 4\lambda)^{(5m-7)/2} + B_3 (5 - 4\lambda)^{5(m-1)/2} \qquad (3.1) \\ h_m &= C_1 (5 - 4\lambda)^{(5m-8)/2} + C_2 (5 - 4\lambda)^{5(m-1)/2} + C_3 (5 - 4\lambda)^{(5m-3)/2} \end{aligned}$$

Here the constants A_1 and A_2 are to be calculated from the initial data (2.4), and the remaining constants are expressed in terms of them by means of the formulas

$$B_1 = (8 - 5m) A_1, \qquad B_2 = \frac{25(1-m)}{2} A_2$$

$$B_3 = \frac{5}{21(5m+2)} [10(m-1)(3-5m) + 9A_2], \qquad C_1 = \frac{36 - 35m}{5} A_1 \qquad (3.2)$$

$$C_2 = \frac{5(3-7m)}{2} A_2, \quad C_3 = \frac{1}{7(5m+2)(3-5m)} [50(m-1)(3-5m) + 3(1-35m) A_2],$$

The asymptotic representations of density and pressure are written as

$$\rho = \rho_{20} (5 - 4\lambda)^{-6/5} [1 + \varepsilon B_3 t^{-m} (5 - 4\lambda)^{5m/2}]$$

$$p = p_{20} (5 - 4\lambda)^{-3/2} [1 + \varepsilon C_3 t^{-m} (5 - 4\lambda)^{5m/2}]$$

It is not difficult to see that as $\lambda \to -\infty$ the product

$$\varepsilon t^{-m} (5 - 4\lambda)^{5m/2} \qquad (3.3)$$

can prove to be a quantity of order unity, even if $t \to \infty$, and the parameter ε is chosen as small as desired. Hence it follows that for large negative values of the coordinate x one cannot seek the solution of the Euler equations in the form (1.3).

In fact, in the region bounded by a vacuum, different terms in the expansion for density and pressure become of comparable order of magnitude, which contradicts the basic principles of perturbation theory. In order to extend the solution properly into this region we use the method of matched inner and outer asymptotic expansions, the essentials of which are explained in the books of Van Dyke [11] and Cole [12].

From the condition that the order of magnitude of the product (3.3) be finite follows the determination of a new similarity variable

$$\eta = -4\varepsilon^{2/(5m)} t^{-2/5} \lambda = -4\varepsilon^{2/(5m)} A^{-2/5} x t^{-1}$$

in the inner region of gas flow. The singularity in the function f_m as $\lambda \to -\infty$ is not so strong as to show an effect on this choice. Thus in the inner region we set

$$\begin{aligned} v &= \frac{1}{4} A^{3/5} \varepsilon^{-2/(5m)} [F_0(\eta) + \varepsilon^{2/(5m)} t^{-2/5} F_1(\eta)] \\ \rho &= \rho_{20} \varepsilon^{1/m} t^{-1} [G_0(\eta) + \varepsilon^{2/(5m)} t^{-2/5} G_1(\eta)] \qquad (3.4) \\ p &= \frac{1}{20} \rho_{20} A^{6/5} \varepsilon^{2/(5m)} t^{-7/5} [H_0(\eta) + \varepsilon^{2/(5m)} t^{-2/5} H_1(\eta)] \end{aligned}$$

We now use the formulas giving the exact solution of the short-blow problem, and the asymptotic representations (3.1) that determine the solution of the perturbation problem to deduce the limiting conditions that the unknown functions must satisfy as $\lambda = \to -1$.

Following the standard procedure in the method of matched outer and inner asymptotic expansions, we substitute all the relations mentioned into Eqs. (1.3), giving the characteristics of the gas in the region whose boundary serves as the front of the strong shock wave. Hence for $\eta \to 0$ we deduce for the leading terms

$$F_0 \to -\eta, \quad G_0 \to \eta^{-5/2} + B_3\eta^{-5(1-m)/2}, \quad H_0 \to \eta^{-3/2} + C_3\eta^{-(3-5m)/2} \tag{3.5}$$

and quite analogously for the functions of the first approximation

$$F_1 \to -2(1 - A_2\eta^{5m/2}), \quad G_1 \to -\left(\frac{25}{2}\eta^{-7/2} - B_3B_4\eta^{-(7-5m)/2}\right)$$

$$H_1 \to -\left({}^{15}/_2\,\eta^{-3/2} - C_3C_4\eta^{-5(1-m)/2}\right) \tag{3.6}$$

The coefficients appearing here are found with the use of Eqs. (3.2), whence

$$B_4 = \frac{25(m-1)}{42(5m-2)}[50(m-1)(3-5m) + 3(1-35m)A_2]$$

$$C_4 = \frac{5}{14(5m+2)}[50(m-1)(5m-3) - (245m^2 - 112m - 39)A_2]$$

Substitution of the relations (3.4) into the original Euler equations (1.1) gives in the zeroth approximation

$$G_0\frac{dF_0}{d\eta} + (\eta + F_0)\frac{dG_0}{d\eta} + G_0 = 0, \qquad G_0(\eta + F_0)\frac{dF_0}{d\eta} = 0$$

$$\frac{7}{5}H_0\frac{dF_0}{d\eta} + (\eta + F_0)\frac{dH_0}{d\eta} + \frac{7}{5}H_0 = 0 \tag{3.7}$$

As for the correction quantities, they satisfy the following system of ordinary differential equations:

$$G_0\frac{dF_1}{d\eta} + (\eta + F_0)\frac{dG_1}{d\eta} + \frac{dG_0}{d\eta}F_1 + \left(\frac{7}{5} + \frac{dF_0}{d\eta}\right)G_1 = 0$$

$$G_0(\eta + F_0)\frac{dF_1}{d\eta} + G_0\left(\frac{2}{5} + \frac{dF_0}{d\eta}\right)F_1 + (\eta + F_0)\frac{dF_0}{d\eta}G_1 = -\frac{4}{5}\frac{dH_0}{d\eta} \tag{3.8}$$

$$\frac{7}{5}H_0\frac{dF_1}{d\eta} + (\eta + F_0)\frac{dH_1}{d\eta} + \frac{dH_0}{d\eta}F_1 + \frac{1}{5}\left(9 + 7\frac{dF_0}{d\eta}\right)H_1 = 0$$

There exist two possibilities for the integration of the second of Eqs. (3.7). First, the function $F_0 = \text{const}$ is its simplest solution, but it contradicts the limiting condition for $\eta \to 0$. Second, if we examine the solution $F_0 = -\eta$, then this condition is automatically satisfied. As solutions of the two remaining equations of the system (3.7) it is then possible to take arbitrary functions

$$G_0 = \Phi(\eta), \qquad H_0 = \Psi(\eta) \tag{3.9}$$

the only restrictions on which are simply that for $\eta \to 0$ they must go over into the expressions (3.5).

It is easy to see that with $F_0 = -\eta$ the system (3.8) turns into algebraic equations. As a result

$$F_1 = \frac{4}{3}\frac{1}{\Phi}\frac{d\Psi}{d\eta}, \quad G_1 = -\frac{5}{2}\frac{d\Phi F_1}{d\eta}, \quad H_1 = -\frac{5}{2}\left(\frac{d\Psi}{d\eta}F_1 + \frac{7}{5}\Psi\frac{dF_1}{d\eta}\right) \tag{3.10}$$

that is, the correction to the speed in the first approximation also includes arbitrary functions. As one can convince himself by direct verification, the functions F_1, G_1 and H_1 satisfy the limiting conditions (3.6).

Formulas (3.9) and (3.10) complete the construction of the solution in the inner region, where the basic dependence $v = x/t$ of the speed on the coordinate turns out to be linear. The arbitrariness in the choice of the functions G_0 and H_0 determining the density and pressure of the gas reflects different possibilities of action on it in the initial time period. With a change in the character of the impulse these functions will also change. It is remarkable that the method of matched outer and inner asymptotic expansions, which is essentially a method of analytic continuation of the solution from one region into another, allows consideration of the arbitrariness, which is evident from a physical point of view.

The divergence of the energy integral in the self-similar solution of the short-blow problem is connected [1, 2] with a singularity in the behavior of the gas parameters for $x, \lambda \to -\infty$. The solution constructed in the inner region allows this difficulty to be eliminated. In fact, the functions Φ and Ψ can be chosen, for example, in such a way that they vanish at a finite value of η. Then in the process of expansion the gas in a finite interval of time will occupy only a finite part of the initially empty space. If it is assumed that in a small neighborhood of the boundary separating it from the vacuum a flow of the type of a simple Reimann wave is achieved, then the speed v_b of the edge particles is easily calculated

$$v_b = -\frac{2}{\varkappa - 1}, \qquad a_b = -5a_b \tag{3.11}$$

where a_b is the speed of a sound wave at the initial instant of time in the application of the impulse. For a different nature of the flow near the boundary, Eq. (3.11) cannot be used to determine the speed v_b, but it will remain finite for finite values of the perturbation pressure. In any case, convergence of the energy integral can be postulated for the choice of the functions Φ and Ψ and the construction of the full solution in the outer and inner regions.

4. We show in conclusion how, by use of variations of the basic solution, to consider an exterior counter-pressure p_1, which up to now was assumed negligibly small. Let a_1 be the speed of sound in the undisturbed state. Expanding in series the exact Hugoniot relations [7]

$$v_2 = \frac{5}{6}\left(1 - \frac{a_1^2}{c^2}\right)c, \quad \rho_2 = 6\rho_1\left(1 + 5\frac{a_1^2}{c^2}\right)^{-1}, \quad p_2 = \frac{5}{6}\rho_1 c^2\left(1 - \frac{1}{7}\frac{a_1^2}{c^2}\right) \tag{4.1}$$

which in the case under consideration must replace Eqs. (1.2), it is easy to show that the power-law exponent $m = -4/5$. We introduce the dimensionless time $\tau = t / t_1$, where

$$t_1 = \left(\frac{\rho_1 A^{6/5}}{p_1}\right)^{5/4}$$

If we put the small parameter

$$\varepsilon = t_1^{-4/5} = \frac{p_1}{\rho_1 A^{6/5}}$$

then the expansions (1.3) for the unknown functions in the outer flow region can be written as

$$v = v_{20}\,[f(\lambda) + \tau^{4/5} f_{-4/5}(\lambda)], \qquad \rho = \rho_{20}\,[g(\lambda) + \tau^{4/5} g_{-4/5}(\lambda)]$$
$$p = p_{20}\,[h(\lambda) + \tau^{4/5} h_{-4/5}(\lambda)] \tag{4.2}$$

with the previous values of v_{20}, ρ_{20} and p_{20}. Having taken the dimensionless coordinate λ_s of the shock-wave front in the form

$$\lambda_s = 1 - k\tau^{4/5} \tag{4.3}$$

we have for the speed of its propagation

$$c = 3/5\,\sqrt{p_1/\rho_1}\,\tau^{-2/5}(1 - 7/3\,k\tau^{4/5}) \tag{4.4}$$

The constant k in formula (4.3) remains at present undetermined; its value will be established later. Using the relations (4.1) and (4.4) we deduce the initial conditions for the unknown functions, namely for $\lambda = 1$:

$$f_{-4/5} = -{}^{35}/_9 - {}^{1}/_3 k, \quad g_{-4/5} = -{}^{175}/_9 + 10k, \quad h_{-4/5} = -{}^{5}/_9 + \frac{4}{3}k \tag{4.5}$$

These perturbation functions obviously satisfy the system of linear equations (1. 5), where it is necessary to take $m = -{}^4/_5$. As above, it is very convenient to change to one second-order differential equation for $f_{-4/5}$. Recalling Eq. (2. 2), we have

$$\xi(1-\xi)\frac{d^2 f_{-4/5}}{d\xi^2} - \frac{1}{2}(3 + 13\xi)\frac{df_{-4/5}}{d\xi} - 7 f_{-4/5} = 0 \tag{4.6}$$

We divide the solution of Eq. (4. 6) into two parts

$$f_{-4/5} = \varphi_1 + k\varphi_2$$

and represent the remaining functions in analogous form

$$g_{-4/5} = \psi_1 + k\psi_2, \qquad h_{-4/5} = \chi_1 + k\chi_2$$

The initial values for the quantities introduced in this way follow from (4. 5). They give the possibility of formulating a Cauchy problem for the functions φ_1 and φ_2. For $\xi = {}^1/_7$ we find

$$\varphi_1 = -{}^{35}/_9, \quad d\varphi_1/d\xi = -{}^{490}/_{27}, \quad \varphi_2 = -1/3, \quad d\varphi_2/d\xi = -{}^{49}/_2 \tag{4.7}$$

We write the lineary independent integrals $f^1_{-4/5}$ and $f^2_{-4/5}$ of Eq. (4. 6) as

$$f^1_{-4/5} = \xi^{5/2}(1-\xi)^{-7}(1 + {}^5/_7\,\xi), \qquad f^2_{-4/5} = \sum_{i=2}^{7} b_i(1-\xi)^i$$

Here the coefficients are

$$b_2 = \frac{7}{24\cdot 64}, \quad b_3 = \frac{35}{6\cdot 64}, \quad b_4 = -\frac{35}{32}, \quad b_5 = \frac{35}{12}, \quad b_6 = -\frac{35}{12}, \quad b_7 = 1$$

Both functions $f^1_{-4/5}$ and $f^2_{-4/5}$ are non-regular at the point $\xi = 1$, but their linear combination

$$\mathring{f}_{-4/5} = \frac{7}{12} f^1_{-4/5} - f^2_{-4/5} \tag{4.8}$$

is, as is easily seen, a regular function at that point. Solution of the Cauchy problem (4. 7) allows the functions φ_1 and φ_2 to be represented in the form

$$\varphi_1 = d_1 f^1_{-4/5} + e_1 f^2_{-4/5}, \qquad \varphi_2 = d_2 f^1_{-4/5} + e_2 f^2_{-4/5}$$

with the constants determined in a unique way

$$d_1 = 1.319\cdot 10^3, \quad e_2 = -2.327\cdot 10^3, \quad d_2 = 1.458\cdot 10^2, \quad e_2 = 2.090\cdot 10^2$$

The ratio $d_1 / e_1 = -0.567$, while at the same time $d_2 / e_2 = -0.698$. Both these values are different from the coefficient $-{}^7/_{12} \approx 0.583$, that appears in the expression (4. 8) for $\mathring{f}_{-4/5}$; therefore the functions φ_1 and φ_2 are nonregular at the point $\xi = 1$.

For arbitrary values of the parameter k in the equation of the shock front, the solution of Eq. (4. 6) will also contain a singularity at that point. However, by proper choice of k the flow field can be made to remain analytic at the intersection with the limiting characteristic. The evident condition for this is the requirement

$$\frac{d_1 + kd_2}{e_1 + ke_2} = -\frac{7}{12}$$

from which $k = -1.586$. Finally, we find

$$f_{-4/5} = E\mathring{f}_{-4/5}, \qquad E = -2.658\cdot 10^3 \tag{4.9}$$

For the remaining functions $g_{-4/5}$ and $h_{-4/5}$ the following formulas hold:

$$g_{-4/3} = -5\,(7\xi)^{-7/3}\left\{{}^1\!/_3\,C_0\,(7\xi)^{-1} + E\left[{}^1\!/_2\,(5+7\xi)\,\overset{\circ}{f}_{-4/3} + \xi\,(\xi-1)\,\frac{d\overset{\circ}{f}_{-4/3}}{d\xi}\right]\right\} \qquad (4.10)$$

$$h_{-4/3} = -\frac{1}{7}\,(7\xi)^{-8/3}\left\{5C_0\,(7\xi)^{-1} + E\left[\frac{1}{2}\,(105+203\,\xi)\,\overset{\circ}{f}_{-4/3} + 29\,\xi\,(\xi-1)\,\frac{d\overset{\circ}{f}_{-4/3}}{d\xi}\right]\right\}$$

The constant C_0 is connected with the constant C in the right-hand side of the integral of adiabaticity (2.1) by the relation $C_0 = {}^6\!/_7 C = 0.543\cdot 10^2$

The constructed solution does not need a special extension in the region situated behind the limiting characteristic; therefore the representations (4.2) for the speed, density, and pressure can be used for a description of the entire flow field if one specially selects the corrections that take into account the presence of external counter-pressure.

It is clear that the complete solution of the system of Euler equations in the zone bounded by a vacuum should contain the analytic continuation considered in the previous section. However, terms due to counter-pressure will not have any effect on the structure of the latter. In order to be convinced of this assertion, we extract the asymptotics of the functions $f_{-4/3}$, $g_{-4/3}$ and $h_{-4/3}$ for $\lambda \to -\infty$. It follows from formulas (4.9) and (4.10) that

$$f_{-4/3} = 5.94\cdot 10^2\,(5-4\lambda)^{-2}, \quad g_{-4/3} = -7.26\cdot 10^2\,(5-4\lambda)_{-9/2}, \quad h_{-4/3} = -5.66\cdot 10^2\,(5-4\lambda)^{-7/2}$$

The asymptotic representations of the density and pressure take the form

$$\rho = 6\rho_1\,(5-4\lambda)^{-6/2}\,[1 - 7.26\cdot 10^2\tau^{4/5}\,(5-4\lambda)^{-2}]$$
$$p = {}^3\!/_{10}\,p_1\tau^{-4/5}\,(5-4\lambda)^{-3/2}\,[1 - 5.66\cdot 10^2\tau^{4/5}\,(5-4\lambda)^{-2}] \qquad (4.11)$$

The correction terms from the differences enclosed in brackets on the right-hand sides of the relations (4.11) become arbitrarily small as $\lambda \to -\infty$ for fixed values of time. Consequently the self-similar solution of the short-blow problem gives the leading terms in the expansions of the parameters of the gas also in the region behind the limiting characteristic.

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REDUCTION OF THE THREE-DIMENSIONAL AXISYMMETRIC PROBLEMS OF THE THEORY OF ELASTICITY TO THE BOUNDARY VALUE PROBLEMS FOR THE ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

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Iu. I. SOLOV'EV
(Novosibirsk)
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Expressions for the general solution of the axisymmetric problem obtained earlier in terms of two analytic functions [1, 2] are transformed in such a manner, that only one of these functions remains under the integral sign. This also leads to the possibility of solving the axisymmetric problems by employing the methods used in the solution of the plane problem. Basically, similar transformations were employed in [3 - 5] for the particular case of a plane boundary.

A series solution for a hollow sphere with various boundary conditions at its surface is used to illustrate the method.

1. As was shown in [2], the components of the elastic displacement in an axisymmetric deformation of a solid of revolution can be written as

$$2Gw(z,r)=\frac{1}{\pi i}\int_{\bar t}^{t}[\varkappa\varphi(\zeta)-(2z-\zeta)\varphi'(\zeta)-\psi(\zeta)]\frac{d\zeta}{g(t,\zeta)}$$

$$2Gu(z,r)=-\frac{1}{2\pi ir}\int_{\bar t}^{t}[\varkappa\varphi(\zeta)+(2z-\zeta)\varphi'(\zeta)+\psi(\zeta)]\,g_1(t,\zeta)\,d\zeta \tag{1.1}$$

$$g(t,\zeta)=\sqrt{(\zeta-t)(\zeta-\bar t)},\qquad g_1(t,\zeta)=(2\zeta-t-\bar t)/g(t,\zeta)$$

Here z and r are the cylindrical coordinates (z is the axis of symmetry), w, u are, respectively, the axial and radial displacements of a point, $\varkappa=3-4\nu$, ν is the Poisson ratio, G is the shear modulus, φ and ψ are analytic functions of the complex variable $\zeta=x+iy$, holomorphic in a symmetrical plane region D occupied by the meridional section of the body, x, y are rectangular coordinates lying in the plane of the above cross-section (x-axis coincides with the z-axis), and the points $t=z+ir$ and $\bar t=z-ir$ lie on this plane within D. The order of integration in (1.1) is arbitrary. The analytic functions satisfy the condition